prof. Vidya Sagar
Department of physics
B.Sc (part - I) Hons

Establish relation between different elastic Constant

- According to the basic law in elasticity, namely Hooke's law 'within elastic limit, stress is proportional to strain'.

$$\therefore \quad \text{Stress} \propto \text{Strain}$$

$$\therefore \quad \frac{\text{Stress}}{\text{Strain}} = \text{Constant}$$

This Constant is Called modulus of elasticity or Coefficient of elasticity. Corresponding to the three types of Strain, there are three different moduli of elasticity as follow:

(i) Modulus of longitudinal elasticity or young's Modulus.

$$Y = \frac{\text{Longitudinal stress}}{\text{Longitudinal Strain}}$$

$$= \frac{F/A}{\frac{\Delta l}{l}}$$

$$\boxed{Y = \frac{Fl}{A\Delta l}}$$

(ii) Modulus of volume or bulk elasticity

$$K = \frac{\text{Volume stress}}{\text{Volume strain}}$$

$$= \frac{\Delta P}{\frac{\Delta V}{V}}$$

$$\boxed{K = V\frac{\Delta P}{\Delta V}}$$

(iii) Modulus of Shear elasticity of rigidity

$$\eta = \frac{\text{Shearing Stress}}{\text{Shearing Strain}}$$

$$= \frac{F/A}{\theta}$$

$$\therefore \boxed{\eta = \frac{F}{A \cdot \theta}}$$

Poisson's ratio :- The increase in length of a body in a direction is always accompained by sideways contraction. This sideways contraction is directly proportional to the longitudinal extension. The ratio of lateral strain to the longitudinal ~~extension~~ strain is called poisson's ration. Thus

$$\text{poission's ration} = \frac{\text{Lateral Strain}}{\text{Longitudinal Strain}}$$

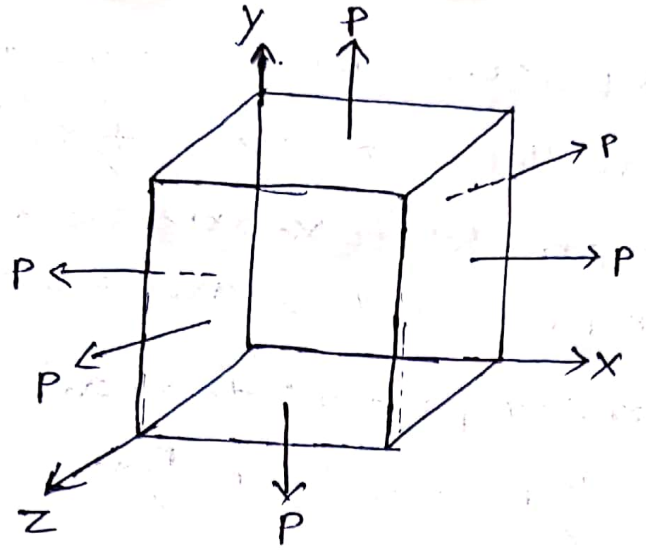
$$\sigma = \frac{\Delta D/D}{\Delta L/L}$$

$$\boxed{\sigma = \frac{L}{D} \cdot \frac{\Delta D}{\Delta L}}$$

The above four constant y, K, η and σ are called elastic constant.

Relation between Elastic Constant :-

Let us consider a cube of length l whole sides are parallel to co-ordinate axes x, y and z respectively. Let a uniform normal stress p be applied over its surface. Then each side of cube is under an extensional stress p.



The Stress p acting parellel to the x-axis will produce extension parellel to x-axis and Compressions parallel to y-axis and z-axis.

$\therefore$ Extension parellel to x-axis = Longitudinal Strain × initial Length

$$= \frac{\text{Stress}}{\text{young's Modulus}} \times \text{initial Length}$$

$$= \frac{P}{Y} \times l$$

Compression parallel to y-axis = Lateral Strain × initial length

= poisson's ratio × Longitudinal Strain × initial length

$$= \sigma \times \frac{P}{Y} \times l$$

Compression parallel to z-axis $= \sigma \times \frac{P}{Y} \times l$

Similarly, the stress P acting parallel to y-axis produces:

extension parallel to y-axis $= \frac{P}{Y} \times l$

Compression parallel to x-axis $= \sigma \times \frac{P}{Y} \times l$

and Compression parallel to z-axis $= \sigma \times \frac{P}{Y} \times l$

Also the Stress P acting parallel to z-axis produces:

extension parallel to z-axis $= \frac{P}{Y} \times l$

Compression parallel to x-axis $= \sigma \times \frac{P}{Y} \times l$

and Compression parallel to z-axis $= \sigma \times \frac{P}{Y} \times l$

Hence, net extension parallel to x-axis

$$= \frac{P}{Y} l - \sigma \frac{P}{Y} l - \sigma \frac{P}{Y} l$$

$$= \frac{P}{Y}(1-\sigma-\sigma) l$$

$$= \frac{P}{Y}(1-2\sigma) l$$

net extension parallel to y-axis

$$= -\sigma \frac{P}{Y} l + \frac{P}{Y} l - \sigma \frac{P}{Y} l$$

$$= \frac{P}{Y}(1-2\sigma) l$$

and net extension parallel to z-axis

$$= -\sigma \frac{P}{Y} l - \sigma \frac{P}{Y} l + \frac{P}{Y} l$$

$$= \frac{P}{Y}(1-2\sigma) l$$

Thus each side of the Cube increase from l to $l + \frac{P}{Y}(1-2\sigma) l$

$\therefore$ The new volume is therefore

$$\left[l + \frac{P}{Y}(1-2\sigma) l\right]^3 = l^3\left[1 + \frac{P}{Y}(1-2\sigma)\right]^3$$

$$= l^3\left[1 + \frac{3P}{Y}(1-2\sigma)\right] \text{ approx}$$

Since the initial volume was l^3,

The change in volume $= l^3\left[1 + \frac{3P}{Y}(1-2\sigma)\right] - l^3$

$$= l^3\left[\cancel{1} + \frac{3P}{Y}(1-2\sigma) - \cancel{1}\right]$$

$$= \frac{3P}{Y}(1-2\sigma) l^3$$

Hence,

$$\text{volume strain} = \frac{\text{Change in volume}}{\text{initial volume}}$$

$$= \frac{\frac{3P}{Y}(1-2\sigma)l^3}{l^3}$$

$$= \frac{3P}{Y}(1-2\sigma)$$

Now, considering the cube as a whole, its surface being under a uniform normal stress P undergoes the volume strain $\frac{3P}{Y}(1-2\sigma)$. Its bulk modulus is therefore

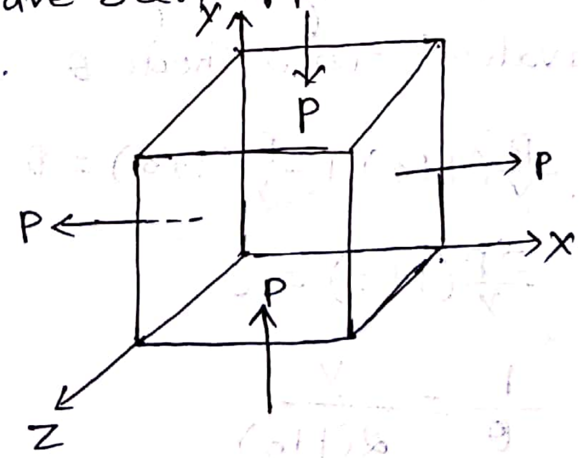
$$K = \frac{\text{normal stress}}{\text{volume strain}}$$

$$= \frac{P}{\frac{3P}{Y}(1-2\sigma)}$$

$$K = \frac{Y}{3(1-2\sigma)}$$

or, $$\boxed{Y = 3K(1-2\sigma)} \quad \text{—— (1)}$$

Let us we consider a cube of side l, upon an extensional stress P to x-axis, and a compressional stress P parallel to y-axis have been applied. There is no stress along the z-axis.



The extensional stress P produces

$$\text{extension along x-axis} = \frac{P}{Y} \times l$$

$$\text{Compression along y-axis} = \sigma \times \frac{P}{Y} \times l$$

$$\text{and Compression along z-axis} = \sigma \times \frac{P}{Y} \times l$$

The Compressional Stress p produces

Compression along y-axis $= \frac{p}{Y} \times l$

extension along x-axis $= \sigma \times \frac{p}{Y} \times l$

extension along z-axis $= \sigma \times \frac{p}{Y} \times l$

Thus,

net extension along x-axis $= \frac{p}{Y} l + \frac{\sigma p}{Y} \times l$

$$= \frac{p}{Y}(1+\sigma) l$$

and net Compression along y-axis $= \frac{\sigma p}{Y} l + \frac{p}{Y} l$

$$= \frac{p}{Y}(1+\sigma) l$$

There are equal extension and Compression along z-axis. Hence there is no change along z-axis.

Since, each side of the cube is of initial length l, the extensional and Compressional Strain along the x- and y-axes are each equal to

$$\frac{\frac{p}{Y}(1+\sigma) l}{l} = \frac{p}{Y}(1+\sigma)$$

Now, Simultaneous equal extensional and Compressional Strains at right angle to each other, are ~~equal~~ equivalent to a Shear θ

$$\therefore \frac{p}{Y}(1+\sigma) + \frac{p}{Y}(1+\sigma) = \theta$$

or, $\frac{2p}{Y}(1+\sigma) = \theta$

or, $\frac{p}{\theta} = \frac{Y}{2(1+\sigma)}$

Further, the extensional Stress p parallel to the x-axis and the Compressional Stress p parallel to the y-axis will be equivalent to a Shearing stress p. Hence the modulus of rigidity of the meterial of the Cube is

$$\eta = \frac{\text{Shearing Stress}}{\text{Shear}}$$

$$\eta = \frac{P}{\theta}$$

Putting the value of $\frac{P}{\theta}$ ~~from~~ in equation

$$\frac{P}{\theta} = \frac{Y}{2(1+\sigma)}$$

$$\eta = \frac{Y}{2(1+\sigma)}$$

or, $\boxed{Y = 2\eta(1+\sigma)}$ ——— (2)

or, $\frac{Y}{2\eta} = 1+\sigma$

or, $\sigma = \frac{Y}{2\eta} - 1$

Let us put this value of σ in eqn (1), then

$$Y = 3K\left[1 - 2\left(\frac{Y}{2\eta} - 1\right)\right]$$

$$= 3K\left(3 - \frac{Y}{\eta}\right)$$

or, $Y = 9K - \frac{3KY}{\eta}$

or, $Y + \frac{3KY}{\eta} = 9K$

or, $Y\left(1 + \frac{3K}{\eta}\right) = 9K$

or, $Y\left(\frac{\eta + 3K}{\eta}\right) = 9K$

$\therefore \boxed{Y = \frac{9\eta K}{\eta + 3K}}$ ——— (3)

from equation 1 and 2 give

$$3K(1-2\sigma) = 2\eta(1+\sigma)$$

or, $3K - 6K\sigma = 2\eta + 2\eta\sigma$

or, $3K - 2\eta = 6K\sigma + 2\eta\sigma$

or, $3K - 2\eta = (6K + 2\eta)\sigma$

or, $$\boxed{\sigma = \frac{3K - 2\eta}{6K + 2\eta}} \quad \text{——— (4)}$$

Equation 1, 2, 3 and 4 are the various relation among the elastic Constant.